Operator quantization of constrained WZNW theories and coset constructions.

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Abstract

Using two WZNW theories for Lie algebras g and h, $h \subset g$, we construct the associative quotient algebra which includes a class of g/h coset primary fields and currents.

1. In a recent article [1] for generic g/h coset conformal field theory [2] a class of Virasoro primary fields and currents was constructed.They are explicitly expressed in terms of two Wess-Zumino-Novikov-Witten (WZNW) theories [3, 4] for Lie algebras g and $h \subset g$.

It turns out that these fields satisfy the usual definition of primary fields only in the weak sense, i.e. inside correlation functions. The object of this paper is to show that outside correlation functions the primary fields can be represented by elements of an associative quotient algebra Ω/Υ. Since coset fields take values in a quotient algebra, the g/h theory is gauge invariant.The gauge transformations are generated by the ideal Υ.

To obtain Ω/Υ we use broken affine primary fields of the WZNW theory for g and an auxiliary WZNW theory for h. Elements of the ideal can be treated as first class constraints and our construction as quantization of the constrained WZNW theory for $g \oplus h$. In the case of Abelian cosets this construction was obtained in [5] using a version of the generalized canonical quantization method [6].

States of coset theories are often identified with BRST invariant states of gauged WZNW actions [7, 8]. The algebra Ω/Υ is also BRST invariant. The corresponding state space is equivalent to the ghost free sector of the BRST approach. However in the full BRST invariant state space which includes ghosts, the fields under consideration may not satisfy the definition of primary fields even in the weak sense.

In what follows we treat only the holomorphic part.

2. We begin with the symmetry algebra of the WZNW theory for g [3, 4]

$$[J^a(m), J^b(n)] = if^{abc}J^c(m+n) + km\delta^{ab}\delta_{m+n,0}, \tag{1}$$

$$[L^g(m), L^g(n)] = (m-n)L^g(m+n) + c^g\left[\frac{1}{12}(m^3-m)\right]\delta_{m,-n}, \tag{2}$$

$$[L^g(m), J^a(n)] = -nJ^a(m+n), \qquad c^g = \frac{2k\,dim\,g}{2k+Q_g}.$$

Here $m, n \in Z$, f^{abc} are the structure constants of g, k and c^g are the central charges and Q_g is the quadratic Casimir in the adjoint representation of g. The Virasoro generator $L^g(m)$ is given by Sugawara construction

$$L^g(m) = \frac{1}{2k+Q_g}\sum_n : J^a(m-n)J^a(n) :.$$

Let $G_R(z)$ be the WZNW primary field

$$[J^a(m), G_R(z)] = z^m G_R(z)t^a_R,$$

$$[L^g(m), G_R(z)] = z^{m+1}\partial_z G_R(z) + \Delta_R(m+1)z^m G_R(z),$$

$$[t^a_R, t^b_R] = if^{abc}t^c_R, \qquad \Delta_R = \frac{Q_R}{2k+Q_g}.$$

Here t^a_R is the matrix irreducible representation of the generators of g for the field $G_R(z)$, Δ_R is the conformal dimension of $G_R(z)$ and Q_R is the quadratic Casimir of g in the representation R.

We shall use the ground state $|0\rangle_g$ of the WZNW theory for g

$$ {}_g\langle 0|J^a(m \leq 0) = J^a(m \geq 0)|0\rangle_g = 0. \tag{3}$$

Let $\hat{h}_k$ be a subalgebra of $\hat{g}_k$. One can choose the basis for $\hat{g}_k$, where $\hat{h}_k$ is generated by $J^A(m)$, $A = 1 \ldots dim\, h$, $m \in Z$. Coset Virasoro generators are given by [2]

$$K(m) = L^g(m) - L^h(m), \qquad m \in Z. \tag{4}$$

One can check [2] that $K(m)$ commutes with $J^A(n)$

$$[K(m), J^A(n)] = 0 \tag{5}$$

and satisfy Virasoro algebra (2) with the central charge $c^{g/h} = c^g - c^h$. The coset primary fields $\phi_i(z)$ are defined by the equations

$$[K(m), \phi_i(z)] = z^{m+1}\partial_z \phi_i(z) + \Delta_i(m+1)z^m \phi_i(z), \tag{6}$$

where Δ_i is the conformal dimension of $\phi_i(z)$.

To construct coset fields we extend the operator algebra of the WZNW theory. Let us decompose each primary field G_R in the set of some irreducible representations of h

$$G_R(z) = \sum_l G_{Rl}(z) = \sum_l P_l G_R(z). \tag{7}$$

Here $G_{Rl}(z)$ belongs to the l's representation and P_l is the corresponding projector. The field G_{Rl} satisfies the equation

$$[J^A(m), G_{Rl}(z)] = z^m G_{Rl}(z) t_l^A, \tag{8}$$

where t_l^A is the representation of the generators of h for the field $G_{Rl}(z)$ As well as $G_R(z)$, the field $G_{Rl}(z)$ is the primary field of Virasoro algebra (2)

$$[L^g(m), G_{Rl}(z)] = z^{m+1}\partial_z G_{Rl}(z) + \Delta_R(m+1)z^m G_{Rl}(z). \tag{9}$$

$G_{Rl}(z)$ are called broken affine primary fields [9]. It is convenient to include in the extended operator algebra all the components $G_{Rl}^{\alpha_l}(z)$ of $G_{Rl}(z)$.

To construct coset currents we shall use the field $J(z) = (J^i(z)), J^i(z) = \sum_m J^i(m) z^{-m-1}, i = dim\, h + 1 \ldots dim\, g$. It can be decomposed in the set of some irreducible representations of h

$$J(z) = \sum_s J_s(z).$$

The field $J_s(z)$ satisfies equation (8) for some t_s^A and (9) with the conformal dimension $\Delta = 1$. As well as $G_{Rl}(z)$, $J_s(z)$ is a broken affine primary field.

It follows from (9), (8) and similar equations for $J_s(z)$ that

$$\begin{aligned}
[K(m), G_{Rl}(z)] &= z^{m+1}\left(\partial_z G_{Rl}(z) - \frac{2}{2k+Q_h} : J^A(z) G_{Rl}(z) : t_l^A\right) \\
&+ \Delta_{Rl}(m+1) z^m G_{Rl}(z), \\
[K(m), J_s(z)] &= z^{m+1}\left(\partial_z J_s(z) - \frac{2}{2k+Q_h} : J^A(z) J_s(z) : t_l^A\right) \\
&+ \Delta_s(m+1) z^m J_s(z),
\end{aligned} \tag{10}$$

$$\Delta_{Rl} = \Delta_R - \frac{Q_l}{2k+Q_h}, \qquad \Delta_s = 1 - \frac{Q_s}{2k+Q_h}, \tag{11}$$

where $Q_l\,(Q_s)$ is the quadratic Casimir of h in the representation $l\,(s)$.

At this stage we have the operator algebra $\mathcal{B}_g$ generated by $J^A(m), G_{Rl}^{\alpha_l}(z)$, $J_s^{\alpha_s}(z)$ and their modes. We shall denote by $\mathcal{B}_g^{(0)}$ the subalgebra of all the fields of $\mathcal{B}_g$ which commute with $J^A(m)$

$$[J^A(m), \mathcal{B}_g^{(0)}] = 0 \tag{12}$$

The algebra $\mathcal{B}_g$ is extended further introducing an auxiliary WZNW theory. Let $\hat{h}_{k'}$ be the auxiliary affine Lie algebra

$$[\chi^A(m), \chi^B(n)] = i f^{ABC} \chi^C(m+n) + k' m \delta^{AB} \delta_{m+n,0},$$

where $A, B, C = 1 \ldots dim\, h$, $k' = -k - Q_h$.In our approach the value of k' is dictated by the conformal invariance of the g/h theory [1].The same k' gives nilpotence of the corresponding BRST operator [10].

Let Φ_l be the primary field of the WZNW theory for $\hat{h}_{k'}$

$$[\chi^A(m), \Phi_l(z)] = z^m \Phi_l(z) t_l^{*A}, \tag{13}$$

$$\frac{\partial}{\partial z}\Phi_l(z) = \frac{2}{2k' + Q_h} : \chi^A(z)\Phi_l(z) : t_l^{*A}, \tag{14}$$

where $\chi^A(z) = \sum_m \chi^A(m) z^{-m-1}$, $t_l^{*A} = -(t_l^A)^T$ and Q_h is the quadratic Casimir in the adjoint representation of h. The normal-ordering symbol $::$ means that negative modes of the currents are on the left and non-negative on the right. The ground state $|0\rangle_h$ is defined by the relations

$${}_h\langle 0|\chi^A(m \leq 0) = \chi^A(m \geq 0)|0\rangle_h = 0. \tag{15}$$

Let $\mathcal{B}_h$ be the operator algebra generated by $\chi^A(z), \Phi_l^{\alpha_l}(z)$ and their modes. Then the final entended algebra is the direct product $\mathcal{B}_g \otimes \mathcal{B}_h$.

3. In $\mathcal{B}_g \otimes \mathcal{B}_h$ one can construct the operators

$$\mathcal{G}_{Rl}(z) = (G_{Rl}(z), \Phi_l(z)), \quad \mathcal{J}_s(z) = (J_s(z), \Phi_s(z)), \tag{16}$$

where $(\cdot,\cdot)$ is the bilinear form

$$(G_{Rl}(z), \Phi_l(z)) = \sum_{\alpha_l} G_{Rl}^{\alpha_l}(z)\Phi_l^{\alpha_l}(z). \tag{17}$$

Inside correlation functions $\mathcal{G}_{Rl}(z)$ and $\mathcal{J}_s(z)$ are primary fields of the g/h coset conformal field theory [1]. However they do not satisfy equation (6). Instead we have

$$\begin{aligned} [K(m), \mathcal{G}_{Rl}(z)] &= z^{m+1}\left(\partial_z \mathcal{G}_{Rl}(z) + T_{Rl}(z)\right) + \Delta_{Rl}(m+1)z^m \mathcal{G}_{Rl}(z), \\ [K(m), \mathcal{J}_s(z)] &= z^{m+1}\left(\partial_z \mathcal{J}_s(z) + T_s(z)\right) + \Delta_s(m+1)z^m \mathcal{J}_s(z), \end{aligned} \tag{18}$$

$$\begin{aligned} T_{Rl}(z) &= -\frac{2}{2k + Q_h}\sum_{A=1}^{dimh} : \mathcal{J}^A(z)(G_{Rl}(z)t_l^A, \Phi_l(z)) :, \\ T_s(z) &= -\frac{2}{2k + Q_h}\sum_{A=1}^{dimh} : \mathcal{J}^A(z)(J_s(z)t_s^A, \Phi_s(z)) :. \end{aligned} \tag{19}$$

Here $\mathcal{J}^A(z) = \sum_m \mathcal{J}^A(m) z^{-m-1}$, $\mathcal{J}^A(m) = J^A(m) + \chi^A(m)$ and $::$ means that negative modes of $\mathcal{J}^A(z)$ are on the left and non-negative on the right. Deriving these equations we used (10),(14) and the property of bilinear form (17)

$$\left(G_{Rl}(z)t_l^A, \Phi_l(z)\right) + \left(G_{Rl}(z), \Phi_l(z)t_l^{*A}\right) = 0.$$

Using (5),(8) and (13) one can check that coset Virasoro generators (4) and primary fields (16) commute with $\mathcal{J}^A(m)$

$$[\mathcal{J}^A(m), K(m)] = [\mathcal{J}^A(m), \mathcal{G}_{Rl}(z)] = [\mathcal{J}^A(m), \mathcal{J}_s(z)] = 0.$$

From this and equation (18) it follows that

$$[\mathcal{J}^A(m), T_{Rl}(z)] = [\mathcal{J}^A(m), T_s(z)] = 0. \tag{20}$$

Therefore we can consider equations (18) in the subspace Ω of all the fields of $\mathcal{B}_g \otimes \mathcal{B}_h$ which commute with $\mathcal{J}^A(m)$

$$[\mathcal{J}^A(m), \Omega] = 0. \tag{21}$$

It is easy to see that Ω is an algebra with respect to the operator multiplication. In virtue of (12),

$$\mathcal{B}_g^{(0)} \subset \Omega. \tag{22}$$

The algebra Ω can be reduced further. Let us define the subspace $\Upsilon \subset \Omega$ which is spanned by the fields

$$U(z) = \sum_{A=1}^{dim\, h} : \mathcal{J}^A(z) U^A(z) :, \tag{23}$$

where $U^A(z)$ are some operators. It follows from (19) and (20) that

$$T_{Rl},\, T_s \in \Upsilon. \tag{24}$$

Using equation (21) for $X(z) \in \Omega, U(w) \in \Upsilon$, we have

$$X(z)U(w) = \sum_{A=1}^{dim\, h} : \mathcal{J}^A(w) X(z) U^A(w) :.$$

Expanding the right-hand side in powers of $z - w$, one can see that all the coefficients are of the form (23). Therefore for $X \in \Omega, U \in \Upsilon$

$$XU \in \Upsilon. \tag{25}$$

Similarly,

$$UX \in \Upsilon. \tag{26}$$

From (25) and (26) it follows that Υ is an ideal of Ω and the quotient Ω/Υ is an algebra.Let $\{X\} \in \Omega/\Upsilon$ be the coset represented by the field X. Then equations (18) express the fact that $\{\mathcal{G}_{Rl}(z)\}, \{\mathcal{J}_s(z)\}$ are primary fields of the coset Virasoro algebra

$$\begin{aligned} [\{K(m)\}, \{\mathcal{G}_{Rl}(z)\}] &= z^{m+1}\partial_z\{\mathcal{G}_{Rl}(z)\} + \Delta_{Rl}(m+1)z^m\{\mathcal{G}_{Rl}(z)\}, \\ [\{K(m)\}, \{\mathcal{J}_s(z)\}] &= z^{m+1}\partial_z\{\mathcal{J}_s(z)\} + \Delta_s(m+1)z^m\{\mathcal{J}_s(z)\}. \end{aligned}$$

Thus we have constructed the operator algebra Ω/Υ which includes the coset Virasoro generators $\{K(m)\}$, primary fields $\{\mathcal{G}_{Rl}(z)\}$ and currents $\{\mathcal{J}_s(z)\}$.

The elements of $\mathcal{B}_g^{(0)}$ do not depend on auxiliary fields.Therefore, for $X \in \mathcal{B}_g^{(0)}$ $\{X\} \neq 0$ and all the fields of $\mathcal{B}_g^{(0)}$ are in Ω/Υ.

Let us put this in other words.Elements of the ideal Υ can be treated as constraints. If $U \in \Upsilon$ we shall write $U \approx 0$. Since Υ is an algebra, for $U, V \in \Upsilon$ we have

$$UV \approx 0. \tag{27}$$

Therefore the constraints are first class (compare with [11]).

Equations (25) and (26) tell us that all the fields of Ω are first class [11]. From this it follows that the theory is invariant with respect to the left and right gauge transformations

$$\delta^L_{U(z)}X(w) = U(z)X(w) \approx 0, \qquad \delta^R_{U(w)}X(z) = X(z)U(w) \approx 0,$$

where $U \in \Upsilon$, $X \in \Omega$.

In virtue of (24), equations (18) can be written in the form (compare with (6))

$$\begin{aligned} [K(m), \mathcal{G}_{Rl}(z)] &\approx z^{m+1}\partial_z\mathcal{G}_{Rl}(z) + \Delta_{Rl}(m+1)z^m\mathcal{G}_{Rl}(z), \\ [K(m), \mathcal{J}_s(z)] &\approx z^{m+1}\partial_z\mathcal{J}_s(z) + \Delta_s(m+1)z^m\mathcal{J}_s(z) \end{aligned}$$

4. The coset ground state $|0\rangle$ can be taken in the form $|0\rangle = |0\rangle_g \otimes |0\rangle_h$. With respect to the g/h coset Virasoro algebra (4) $|0\rangle$ is the $sl(2, C)$ invariant ground state

$$K(m)|0\rangle = 0, \quad m \geq -1. \tag{28}$$

It follows from (3),(15) and (23) that for $V(z) \in \Upsilon$ the correlation function $\langle V(z) \rangle$ vanishes

$$\langle V(z) \rangle = 0. \tag{29}$$

Taking into account (25),(26) and (29) for $U(z) \in \Upsilon$, $X_i(z) \in \Omega$, one gets the gauge Ward identities

$$\langle U(z) X_1(z_1) \ldots X_N(z_N) \rangle = 0.$$

From this and equation (24) it follows that inside correlation functions equation (18) can be written in the form (6)

$$\begin{aligned} [K(m), \mathcal{G}_{R\,l}(z)] &= z^{m+1} \partial_z \mathcal{G}_{R\,l}(z) + \Delta_{R\,l}(m+1) z^m \mathcal{G}_{R\,l}(z), \\ [K(m), \mathcal{J}_s(z)] &= z^{m+1} \partial_z \mathcal{J}_s(z) + \Delta_s(m+1) z^m \mathcal{J}_s(z). \end{aligned}$$

In virtue of (16), correlation functions of coset primary fields and currents can be expressed in terms of two WZNW theories for Lie algebra g at level k and h at level $k' = -k - Q_h$ [1].For instance,

$$\langle \mathcal{G}_{R_1 l_1}(z_1) \ldots \mathcal{G}_{R_N l_N}(z_N) \rangle$$

$$= \sum_{\alpha_1 \ldots \alpha_N} {}_g\langle G^{\alpha_1}_{R_1 l_1}(z_1) \ldots G^{\alpha_N}_{R_N l_N}(z_N) \rangle_g \;\; {}_h\langle \Phi^{\alpha_1}_{l_1}(z_1) \ldots \Phi^{\alpha_N}_{l_N}(z_N) \rangle_h.$$

Let us compare the present approach with the BRST one [7, 8]. The BRST operator Q can be written in the form

$$Q = \int dz \left(\sum_{A=1}^{dim h} c_A(z) \mathcal{J}^A(z) + \tilde{Q}(z) \right). \tag{30}$$

Here the ghost field $c_A(z)$ and $\tilde{Q}(z)$ commute with Ω. From this and definition (21) it follows that $Q(z)$ commutes with Ω and our construction is BRST invariant. Since we did not use ghost fields the corresponding BRST invariant state space is ghost free, i.e. has not ghost excitations.

Let us now consider equations (18) in the full BRST invariant operator algebra Ω' which is constructed using ghost fields. To divide out the fields $T_{R\,l}$, T_s one must construct the ideal of Ω' which includes these fields and

does not include coset Virasoro generators and primary fields. Υ is not an ideal of Ω' and it is not clear whether such ideal exists. We leave this problem for future investigations.

In conclusion, we have described the reduction which leads from the WZNW theory for $g \oplus h$ to an operator (sub)algebra of the g/h coset conformal field theory. This approach can be used for operator quantization of more general coset constructions and other constrained systems.

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